Estimating committor function using rare event sampling

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Abstract

The committor function presented is the central object of transition path theory.[1] Defined as the probability of reaching the state B before the state A starting from the configuration given in argument, it can be seen as an ideal reaction coordinate in many contexts. As solving the Backward Kolmogorov equation for which the committor is the solution is out of reach for classical methods such as finite elements method, multiple approaches have been proposed in the literature to build approximate committor functions.[2,3,4,5,6,7,8,9] Due to the high dimensionality of the arguments of this function, using neural networks to parameterise it is motivated by success of such models to learn high dimensional functions. In this work we propose an alternative minimization problem suited to an iterative approach using the AMS algorithm. An illustration of such an iterative procedure is presented.

1. The committor function

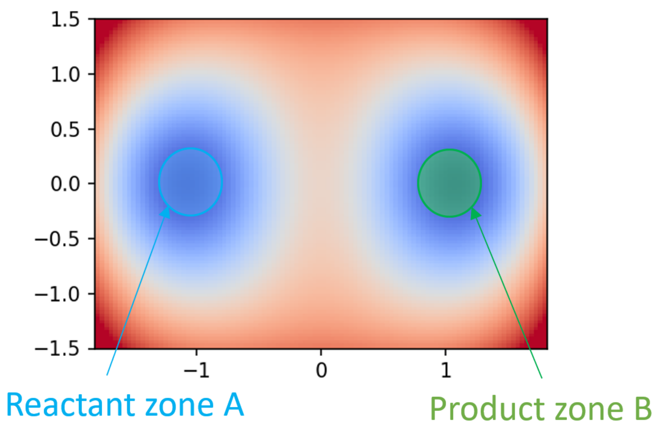

Figure 1: *Illustration of definition of metastable states.*

We consider the overdamped Langevin dynamics:

$$d\mathbf{q}_t = -\nabla V(\mathbf{q}_t)dt + \sqrt{\frac{2}{\beta}}d\mathbf{W}t$$

with infinitesimal generator:

$$\mathcal{L} = -\nabla V\cdot\nabla + \frac{1}{\beta}\Delta$$

$$p_{A\to B}(\mathbf{q}) = \mathbb{P}\left(\mathbf{q}_{\tau_{A\cup B}}\in B\middle|\mathbf{q}_0 = \mathbf{q}\right)$$

$$\tau_X = \inf\left\{t\in(0;+\infty)\,\middle|\,\mathbf{q}_t\in X\right\}$$

The committor function verifies the Backward Kolmogorov equation:

$$\begin{aligned}&\forall\mathbf{q}\in\Omega\backslash(\overline{A}\cup\overline{B}),\quad \mathcal{L}p_{A\to B}(\mathbf{q}) = 0,\\ &\forall\mathbf{q}\in\overline{A},\ p_{A\to B}(\mathbf{q}) = 0,\quad \forall\mathbf{q}\in\overline{B},\ p_{A\to B}(\mathbf{q}) = 1,\end{aligned}\tag{1}$$

with $\overline{A} = A\cup\partial A$ and $\overline{B} = A\cup\partial B$

2. Methods to approximate committor using neural networks

1^{st} method, point-wise approximation

- Multiple MD runs starting from various positions $\mathbf{q}$
- Define bins using a set of collective variables and use "infinitely" long unbiased MD trajectory[7] or multiple weighted trajectories[4]

→ Drawback: CVs parameterizing the committor should be known

2^{nd} method, variational formulation[2,3,6]

$$\underset{f}{\operatorname{arginf}}\left\{\int_{\Omega\backslash(\overline{A}\cup\overline{B})}|\nabla f(\mathbf{q})|^2\,\mathrm{e}^{-\beta V(\mathbf{q})}d\mathbf{q},\ \middle|\ f(\mathbf{q}) = 0,\ \mathbf{q}\in\overline{A},\quad f(\mathbf{q}) = 1,\ \mathbf{q}\in\overline{B}.\right\}\tag{2}$$

→ Drawback: a sampling of the Boltzmann–Gibbs measure is required.

3^{rd} method: fixed point with multiple evaluations[9]

$$\underset{f}{\operatorname{arginf}}\left\{\int_{\Omega\backslash(\overline{A}\cup\overline{B})}\left(\left(I-\mathcal{P}^i\right)f(\mathbf{q}) - \left(\mathcal{P}^b\mathbb{1}_{\overline{B}}\right)(\mathbf{q})\right)^2\mu(d\mathbf{q})\right\}.\tag{3}$$

where μ can be any measure defined on $\Omega\backslash(\overline{A}\cup\overline{B})$ and $\mathcal{P}$ is the propagator:

$$\begin{aligned}\left(\mathcal{P}p_{A\to B}\right)(\mathbf{q}_0) &= \mathbb{E}^{\mathbf{q}_0}\left[p_{A\to B}\left(\mathbf{q}_{t\wedge\tau_{\overline{A}\cup\overline{B}}}\right)\right]\\ &= \mathbb{E}^{\mathbf{q}_0}\left[p_{A\to B}(\mathbf{q}_t)\,\mathbb{1}_{t<\tau_{\overline{A}\cup\overline{B}}}\right] + \mathbb{E}^{\mathbf{q}_0}\left[\mathbb{1}_{\overline{B}}\left(\mathbf{q}_{\tau_{\overline{A}\cup\overline{B}}}\right)\mathbb{1}_{t\geqslant\tau_{\overline{A}\cup\overline{B}}}\right]\\ &= \left(\mathcal{P}^i p_{A\to B}\right)(\mathbf{q}_0) + \left(\mathcal{P}^b\mathbb{1}_{\overline{B}}\right)(\mathbf{q}_0)\end{aligned}$$

The committor verifies:

$$\forall\mathbf{q}\in\Omega\backslash(\overline{A}\cup\overline{B}),\quad\left(I-\mathcal{P}^i\right)p_{A\to B}(\mathbf{q}) - \left(\mathcal{P}^b\mathbb{1}_{\overline{B}}\right)(\mathbf{q}) = 0.$$

→ Drawback: Multiple runs have to be started from the same point.

4^{th} method: fixed point with an ergodic trajectory[5,8]

$$\underset{f}{\operatorname{arginf}}\left\{\frac{1}{2}\int_{\Omega\backslash(\overline{A}\cup\overline{B})}f(\mathbf{q})\left(I-\mathcal{P}^i\right)f(\mathbf{q})\mathrm{e}^{-\beta V(\mathbf{q})}d\mathbf{q} - \int_{\Omega\backslash(\overline{A}\cup\overline{B})}f(\mathbf{q})\mathcal{P}^b\mathbb{1}_{\overline{B}}(\mathbf{q})\mathrm{e}^{-\beta V(\mathbf{q})}d\mathbf{q}\right\}\tag{4}$$

→ Drawback: an ergodic trajectory sampling the Boltzmann–Gibbs measure is required.

3. Proposed minimization problem

Itō formula leads to:

$$dp_{A\to B}(\mathbf{q}_t) = \mathcal{L}p_{A\to B}(\mathbf{q}_t)dt + \sqrt{\frac{2}{\beta}}\nabla p_{A\to B}(\mathbf{q}_t)\cdot d\mathbf{W}_t.$$

Then, $\forall\mathbf{q}_0\in\Omega\backslash(\overline{A}\cup\overline{B})$:

$$p_{A\to B}(\mathbf{q}_t)\mathbb{1}_{t<\tau_{\overline{A}\cup\overline{B}}} + \mathbb{1}_{\overline{B}}(\mathbf{q}_{\tau_{\overline{A}\cup\overline{B}}})\mathbb{1}_{t\geqslant\tau_{\overline{A}\cup\overline{B}}} - p_{A\to B}(\mathbf{q}_0) = \int_0^{t\wedge\tau_{\overline{A}\cup\overline{B}}}\sqrt{\frac{2}{\beta}}\nabla p_{A\to B}(\mathbf{q}_s)\cdot d\mathbf{W}_s$$

Alternative approach:

$$\underset{f}{\operatorname{arginf}}\int_{\Omega\backslash(\overline{A}\cup\overline{B})}\mathbb{E}\left[\left(f(\mathbf{q}_t)\mathbb{1}_{t<\tau_{\overline{A}\cup\overline{B}}} + \mathbb{1}_{\overline{B}}(\mathbf{q}_{\tau_{\overline{A}\cup\overline{B}}})\mathbb{1}_{t\geqslant\tau_{\overline{A}\cup\overline{B}}} - f(\mathbf{q}_0) - \int_0^{t\wedge\tau_{\overline{A}\cup\overline{B}}}\sqrt{\frac{2}{\beta}}\nabla f(\mathbf{q}_s)\cdot d\mathbf{W}_s\right)^2\right]\mu(d\mathbf{q}_0).\tag{5}$$

Using the Euler-Maruyama integration scheme:

$$\mathbf{q}_{n+1} = \mathbf{q}_n - \nabla V(\mathbf{q}_n)\Delta t + \sqrt{\frac{2\Delta t}{\beta}}\mathbf{G}_{n+1},$$

we run K trajectories of length N. Then the discretized loss writes:

$$\mathcal{L}_\theta = \frac{1}{K}\sum_{k=1}^{K}\left(f_\theta\left(\mathbf{q}^k_{N\wedge\lfloor\frac{\tau_{\overline{A}\cup\overline{B}}}{\Delta t}\rfloor+1}\right) - f_\theta(\mathbf{q}^k_0) - \sum_{n=1}^{N\wedge\lfloor\frac{\tau_{\overline{A}\cup\overline{B}}}{\Delta t}\rfloor+1}\sqrt{\frac{2\Delta t}{\beta}}\nabla f_\theta(\mathbf{q}^k_n)\cdot\mathbf{G}^k_n\right)^2,$$

where f_θ is the neural network parameterizing the committor:

$$f_\theta(\mathbf{q}) = \left(1-\mathbb{1}_{\overline{A}}(\mathbf{q})\right)\left[\left(1-\mathbb{1}_{\overline{B}}(\mathbf{q})\right)p_\theta(\mathbf{q}) + \mathbb{1}_{\overline{B}}(\mathbf{q})\right].$$

4. Illustration of the approach combined with AMS

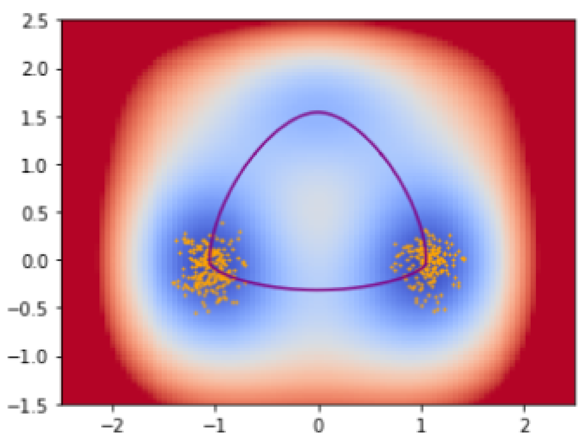

Figure 2: *Train dataset after intitial conditions sampling*

Using a feedforward model made of 3 hidden layers of 20 neurons with tanh activation function. Optimizer: Adam, learning rate 0.001

1. Define: A, Σ_A, B and Σ_B as discs centered on minima.
2. Run short MD starting from minima to gather initial conditions for AMS ($N_{\text{rep}} = 20$).
3. Minimize the loss $\mathcal{L}_\theta$ with trajectories of length $N = 1$

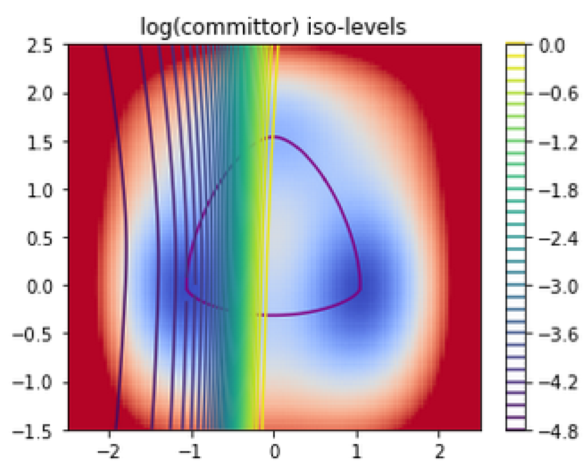

Figure 3: *First committor approximation, after initial conditions sampling*

4. Run AMS forward ($A\to B$) and backward ($B\to A$) and gather all the sub-trajectories of length $N = 1$
5. Re-train the model.

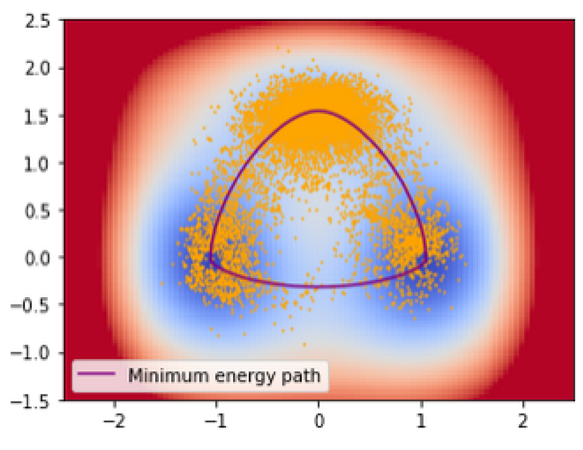

Figure 4: *Train dataset after first AMS*

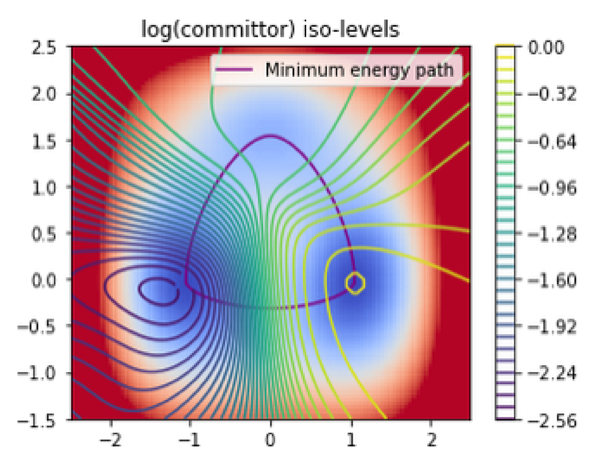

Figure 5: *Second committor approximation, after initial conditions sampling and first AMS*

6. Gather new initial conditions.
7. Run AMS again ($N_{\text{rep}} = 50$).
8. Add the new trajectories (possibly increase the time lagg)

Repeat the step 6 to 9 until the approximate committor function no longer changes.

Table 1: *95% confidence interval of the transition probability obtained after 100 forward and backward AMS runs using various reaction coordinates.*

RC	$\xi(x,y) = x$	NN committor	FE committor
Forward			
$p\pm1.96\,\sigma_p$	$(1.10\pm2.72)\times10^{-3}$	$(4.66\pm2.76)\times10^{-3}$	$(3.78\pm3.11)\times10^{-3}$
Backward			
$p\pm1.96\,\sigma_p$	$(1.38\pm2.82)\times10^{-3}$	$(4.30\pm2.25)\times10^{-3}$	$(3.76\pm3.13)\times10^{-3}$

Conclusion

An alternative approach to train neural networks to approximate the committor function is proposed. This method allows to obtain an approximation leading to more accurate probability estimation using the AMS algorithm.

Nonetheless this approach needs to be studied in more details to better understand why the obtained results differ depending on the choice of the measure μ in equation (5).

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